Spring 2017 MATH5012

Real Analysis II

Exercise 5

(1) Let $f \in L^1(\mathbb{R}^1)$ and $g \in L^p(\mathbb{R}),\ p \in [1, \infty]$.

(a) Show that Young's inequality also holds for $p = \infty$.

(b) Show that equality can hold in Young's inequality when $p = 1$ and ∞, and find the conditions under which this happens.

(c) For $p \in (1, \infty)$, show that equality in the inequality holds only when either f or g is zero almost everywhere.

(d) For $p \in [1, \infty]$, show that for each $\varepsilon > 0$, there exist $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ such that

$$\|f * g\|_p > (1 - \varepsilon)\|f\|_1 \|g\|_p \ .$$

(2) Show that for integrable f and g in $\mathbb{R}^n$, for a.e x,

$$\int f(x - y)g(y)\, dy = \int g(x - y)f(y)\, dy.$$

(3) A family $\{Q_\varepsilon\}$, $\varepsilon \in (0, 1)$ or a sequence $\{Q_n\}_{n \geq 1}$ is called an "approximation to identity" if (a) $Q_\varepsilon,\ Q_n \geq 0$, (b) $\int Q_\varepsilon,\ \int Q_n = 1$, and (c) $\forall \delta > 0$,

$$\int_{|x| \geq \delta} |Q_\varepsilon|(x)\, dx \to 0 \text{ as } \varepsilon \to 0 \text{ or}$$

$$\int_{|x| \geq \delta} |Q_n|(x)\, dx \to 0 \text{ as } n \to \infty.$$

Verify that

(i) $P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$, $x \in \mathbb{R}$; $y \to 0$

(ii) $H_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$, $x \in \mathbb{R}^n$, $t \to 0$,

(iii) $\frac{1}{2\pi} F_k(x) = \begin{cases} \frac{1}{2\pi n} \frac{\sin^2 \frac{kx}{2}}{\sin^2 \frac{x}{2}}, & |x| \leq \pi, \\ 0, & |x| > \pi, \end{cases}$, $x \in \mathbb{R}$, $k \to \infty$

are approximations to identity.

(4) Let f be a continuous function in $\mathbb{R}^n$. Then $f * Q_\varepsilon \to f$ for any approximation to identity Q_ε (uniform in compact sets).

(5) Let $f \in L^1(\mathbb{R}^n)$ and x a Lebesgue point of f. Show that $f * Q_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0$.